Correspondence

A Note on the Phase Velocity in Continuously Loaded Coaxial Cables

The loaded coaxial cable pictured in Fig. 1 has been analyzed from a transmission line viewpoint by Prache[1] and Raisbeck.[2] This approach involves finding the equivalent (static) inductance L and capacitance C per unit length of line and equating the dominant-mode phase velocity to $(LC)^{-1/2}$.

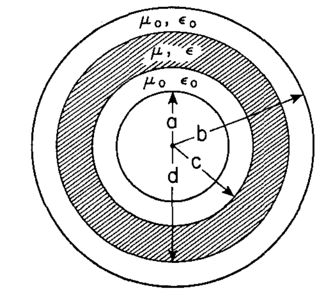

Fig. 1. Cross section of axially uniform, inhomogeneous coaxial cable.

An exact solution for the phase velocity, neglecting losses, can be found for this system by the standard method of generating scalar wave solutions and applying the appropriate boundary conditions at $r=a$, b, c, and d. Thus, the results are a four-by-four determinantal equation, the lowest roots of which determine the propagation constant for the dominant (TM_{01}) mode. This procedure has been carried out by the author to check the accuracy of the transmission line results.

Solutions for the phase velocity were determined with the help of a computer for frequencies in the 100 to 200 megacycle range with dielectric constants ranging from 2 to 10 and relative permeabilities ranging from 2 to 40.[3] For all the cases studied, the computer results agreed with those predicted by the approximate theory to within 2.5 per cent. Part of this variance is due to accuracy limitations on the Bessel function programs used in the computation. One may conclude, therefore, that even in cases of heavy loading, the approximate solution is quite accurate.

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[2] Raisbeck, G., Attenuation in continuously loaded coaxial cables, *Bell Sys. Tech. J.*, vol 37, no 2, Mar 1958.

[3] Large, D. B., Propagation characteristics of an inhomogenous coaxial cable, UCRL-11544, Lawrence Radiation Lab., University of California, Jul 17, 1964.

An Extension to the High Loss Region of the Solution of the Confocal Fabry-Perot Resonator Integral Equation

Recently two papers have appeared that relate the fields in confocal Fabry-Perot resonators to oblate spheroidal coordinates. Zimmerer [1] states "... the spheroidal surfaces within the resonator are surfaces of constant phase and the hyperboloids are surfaces of constant amplitude." Vainshtein [2] shows that starting from an oblate spheroidal resonator and assuming that the propagation is directed largely along the z axis (see Fig. 1), one can determine the amplitude distribution along a resonator plate. The result he obtained agrees very well with that obtained previously by Goubau and Schwering [3] who derived this result more directly from Maxwell's equations.

In the course of their work, the Goubau-Schwering integral (22) appears as

$$\mathcal{E}_\nu^{\pm}(\rho, z_0) e^{jkz_0(1+\rho^2/4z_0^2)} = \frac{ik}{2z_0}\int_0^R \mathcal{E}_\nu^{\pm}(r, -z_0) \cdot e^{-jkz_0(1+r^2/4z_0^2)} J_\nu\left(\frac{kr\rho}{2z_0}\right) r\,dr \quad (1)$$

It is the purpose of this note to show that the Goubau-Schwering equation can be related to a known integral equation which appears when solving for oblate spheroidal wave functions. Then, as a result, eigenvalues for the high loss regions can be related to tabulated values which will extend the eigenvalue results previously published [4].

Consider, from Fig. 1, that for $R \ll Z_0$

$$\rho \sim z_0\theta' \qquad r \sim z_0\theta \qquad \rho d\rho \sim z_0^2\theta d\theta$$

$$\theta_{\max} \sim \frac{R}{z_0} = \frac{a}{\sqrt{kz_0/2}} \quad (2)$$

where a is the Goubau-Schwering parameter defined by

$$a = \sqrt{\frac{k}{2z_0}}\,R \quad (3)$$

Since the oblate spheroidal coordinates account for the phase terms, one can define

$$\mathcal{E}_\nu^{\pm}(r, -z_0)e^{jkz_0(1+r^2/4z_0^2)} = G_\nu(\theta, -z_0)$$

$$\mathcal{E}_\nu^{\pm}(\rho, z_0)e^{-jkz_0(1+\rho^2/4z_0^2)} = G_\nu(\theta', z_0) \quad (4)$$

Also because a reiterative system is desired, one can write

$$G_\nu(\theta', z_0) = pG_\nu(\theta, -z_0) \quad (5)$$

where p is the eigenvalue and represents the ratio of the field intensity amplitude at a phase transformer to the succeeding phase transformer, i.e., it is a measure of the loss.

Now substituting (2) to (5) into (1) leads to

$$pG_\nu(\theta' - z_0) = j\frac{kz_0}{2}\int_0^{a/\sqrt{kz_0/2}} G_\nu(\theta, -z_0) \cdot J_\nu\left(\frac{kz_0}{2}\theta\theta'\right)\theta d\theta \quad (6)$$

A change of variables

$$\theta = \frac{2a\phi}{\pi\sqrt{kz_0/2}} \qquad \theta' = \frac{2a\phi'}{\pi\sqrt{kz_0/2}} \quad (7)$$

in (6) results in

$$pG_\nu(\phi', -z_0) = j\frac{4a^2}{\pi^2}\int_0^{\pi/2} G_\nu(\phi, -z_0) \cdot J_\nu\left(\frac{4a^2}{\pi^2}\phi\phi'\right)\phi d\phi \quad (8)$$

The angle θ was assumed to be small in the above work, and the angle ϕ was defined only for $-\pi/2 \le \phi \le \pi/2$. As a result of this latter condition, ϕ may be assumed to be periodic outside of this range, and hence one can express ϕ in a Fourier series and find its first coefficient to be $4/\pi$. With this (8) becomes

$$pG_\nu(\phi', -z_0) = j\frac{16a^2}{\pi^3}\int_0^{\pi/2} G_\nu(\phi, -z_0) \cdot J_\nu\left(\frac{64a^2}{\pi^4}\sin\phi\sin\phi'\right)\sin\phi d\phi \quad (9)$$

One may now compare (9) with (5), (3), and (12) of Flammer [5]. The integral equation that appears in Flammer is in a slightly different form than that used in this work. The derivation of the desired relation follows immediately. Flammer states

$$S_{mn}(-ic, \eta)R_{mn}^{(1)}(-ic, i\zeta) = \frac{i^{m-n}}{2}\int_0^\pi e^{ic\eta\zeta\cos\theta_0}J_m\left[c(1-\eta^2)^{1/2}(\zeta^2+1)^{1/2} \cdot \sin\theta_0\right]S_{mn}(-ic, \cos\theta_0)\sin\theta_0 d\theta_0 \quad (10)$$

Let $\zeta = 0$ and $\eta = \cos\theta_0'$ and divide the range of the integration into two parts, from $\theta_0 = 0$ to $\theta_0 = \pi/2$ and from $\theta_0 = \pi/2$ to $\theta_0 = \pi$. In the latter range of integration, let $\theta_0 = \pi - \psi_0$ and for $(n-m)$ even, one has

$$S_{mn}(-ic, -\cos\psi) = S_{mn}(-ic, \cos\psi) \quad (11)$$

Therefore (10) can be written in the form

$$S_{mn}(-ic, \cos\theta_0')R_{mn}^{(1)}(-ic, io) = \frac{i^{m-n}}{2}\int_0^{\pi/2} J_m[c\sin\theta_0'\sin\theta_0] \cdot S_{mn}(-ic, \cos\theta_0)\sin\theta_0 d\theta_0 \quad (12)$$

The desired result may now be obtained by comparing (9) with (12) and relating

$$a \to \pi^2/8\sqrt{c}$$

$$p \to \pi/4cR_{mn}^{(1)}(-ic, io)$$

$$G_\nu(\phi, -z_0) \to S_{mn}(-ic, \cos\theta_0) \quad (13)$$

where $R_{mn}^{(1)}(-ic, io)$ and $S_{mn}(-ic, \cos\theta_0)$ are respectively the oblate spheroidal radial and angular wave functions. Because the values of $R_{mn}^{(1)}(-ic, io)$ are tabulated for various values of m and n [6], it is a simple

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